Advanced Calc
MAA4102 & MAA5104

Prac-C

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Please. Use "$f(x)$ notation" when writing fncs; in particular, for trig and log fncs. E.g, write "$\sin(x)$" rather than the horrible $\sin x$ or $[\sin x]$. Do ***not*** approx.: If your result is "$\sin(\sqrt{\pi})$" then write that rather than .9797⋯. Write expressions unambiguously e.g, "$1/a+b$" should be bracketed either $[1/a]+b$ or $1/[a+b]$. (Be careful with **negative** signs!)

Abbrevs: **seq** for "sequence". Use **nv-** for "non-void", e.g "consider a nv-closed set K". Use **MS** for "metric space".

ITOf for "in terms of". **st.** for "such that" **poly** for "polynomial" **coeff** for "coefficient"

Use $\overline{\mathbb{R}}$ for $[-\infty,+\infty]$, the "extended reals". Use **LUBP** for "(the) LUB-property". Use **GLBP** for "(the) greatest lower bound property".

For each of the limit questions, write "$+\infty$", "$-\infty$", a real number, or *–if none of these–* "DNE". Write **DNE** in a blank <u>if</u> the described object does not exist or if the indicated operation cannot be performed. For the **True/False** questions, circle *T* or *F*, **<u>but</u>** *write in* "*DNE*" if the question is inconsistent.

C0: *Greek*: Write the **other case** and name.
Eg: "α: ______ B: ______." You fill in: <u>A *alpha*</u> <u>β *beta*</u>.

Γ: ______ ρ: ______ δ: ______

C1: Show no work.

a) Ordered set $(\mathbb{Q},<)$ has LUBP. *T* *F*

Ordered set $\mathbb{Q}\cap[3,5]$ has LUBP *T* *F*

Half-open real-interval $(3,5]$ has LUBP. *T* *F*

The Integers $\mathbb{Z}$ has LUB-prop. *T* *F*

b) The plane $\mathbb{R}\times\mathbb{R}$ has the LUB-prop. *T* *F*

c) If ordered set $(\mathbb{T},<)$ has the LUB-property, then it has the GLB-property. *T* *F*

d) Using the floor and ceiling fncs: $\lfloor\lceil\pi\rceil\rfloor=$ ______ and $\lceil\sqrt{19}\,\rceil=$ ______.

e) $\lim_{x\to0^+}\cos(\frac{1}{x})=$ ______. $\lim_{x\to\infty}\cos(\frac{1}{x})=$ ______.

$\lim_{x\to0^+}x\cdot\cos(\frac{1}{x})=$ ______. $\lim_{x\to\infty}x\cdot\cos(\frac{1}{x})=$ ______.

f) $\lim_{x\to0}\frac{\sqrt{9+x}-\sqrt{9-x}}{x}=$ ______.

g) Let $F(x):=\sin(x)^x$. Its derivative, then, is $F'(x)=$ ______.

h) An example $h\colon(0,1)\to\mathbb{R}$ of a **bnded, cts** fnc which is NOT *uniformly*-continuous is $h(x):=$ ______.

i) On $\mathbb{Z}_+$, write $x\ \$\ y$ IFF $xy<0$. So $\$$ is [Circle]
Transitive: *T* *F*. **Symm.**: *T* *F*. **Reflex.**: *T* *F*.

On $\mathbb{Z}$, say that $x\nabla y$ IFF $x-y\leqslant1$. Then ∇ is:
Trans.: *T* *F*. **Symm.**: *T* *F*. **Reflex.**: *T* *F*.
(Be *careful* on <u>both</u> parts!)

j) Let $\Omega:=\{+\infty,-\infty\}\cup\mathbb{Q}$, equipped with the usual arctan-metric atm. Then each atm-Cauchy sequence converges in Ω: *T* *F*

C2: Our space is $\mathbb{R}$ with the usual Euclidean metric $\mathsf{d}(x,z):=|x-z|$.

I) These *closed* bnded nv-intervals $A_n:=$ ______, when unioned, form a set $\bigcup_{n=1}^{\infty}A_n=$ ______ which is not closed.

II) Give an example of a set,

$$\Big\{\ \_\_\_\ \in\mathbb{R}\ \Big|\ \_\_\_\_\_\_\_\_\_\_\ \Big\},$$

which has *exactly one* accumulation point in $\mathbb{R}$.

III) Suppose that $U,V_1,V_2,\dots$ are open sets of $\mathbb{R}$, and $E,K_1,K_2,\dots$ are closed sets. **Circle** those of the following sets which are <u>guaranteed</u> to be *closed* in $\mathbb{R}$.

$E\smallsetminus U$. $U\smallsetminus E$. $K_1\smallsetminus E$. $\bigcap_{n=1}^{\infty}K_n$.

$\mathbb{R}\smallsetminus\big[\bigcup_{n=1}^{\infty}V_n\big]$. $E\cup K_1$. $E\cap K_1$.

Essays. On your own sheets of lined paper, give the following definitions or proofs. No "scratch work" accepted, only complete, grammatical, coherent sentences. Write **every 2nd or every 3rd line** for math essays.

C2: [A] Write the *definition* of "$\lim_{x\to 4} x^2 = 5$". [*Hint:* You need 3 quantifiers, an ε and a δ.]

[B] Given seqs. with $\lim(\vec{\mathbf{b}}) = 2$ and $\lim(\vec{\mathbf{c}}) = 4$, let $\vec{\mathbf{a}} := \vec{\mathbf{b}} \cdot \vec{\mathbf{c}}$. Given $\varepsilon>0$, produce –with careful proof– a posint N st.: $\forall k \geqslant N$: $|a_k - 8| < \varepsilon$.

[C] Given seqs. with $\lim(\vec{\mathbf{b}}) = 2$ and $\lim(\vec{\mathbf{c}}) = 4$, let $\vec{\mathbf{s}} := \vec{\mathbf{b}} + \vec{\mathbf{c}}$. Given $\varepsilon>0$, produce –with careful proof– a posint N st.: $\forall k \geqslant N$: $|s_k - 6| < \varepsilon$.

[D] A sequence $\vec{\mathbf{c}} \subset \mathbb{R}$ is ***Cauchy*** IFF ... [*Hint:* Be precise with your quantifiers]

C3: State the Nested Intervals Thm. Prove it.
State the Bolzano-Weierstrass Thm. Prove it.
State the Heine-Borel Thm. Prove it.

C4: On a set Ω, a fnc $\mathsf{m}{:}\Omega\times\Omega\to[0,\infty)$ is a ***metric*** if: $\forall P, Q, R \in \Omega$: (*Write the 3 remaining axioms.*)

C5: Consider a MS (Ω, m), and points $P, Q \in \Omega$.

[i] The radius-5 (open) ***ball*** is

$$\text{Ball}_5(P) := \{ \_\_\_ \in \Omega \mid \_\_\_\_\_\_\_\_\_\_\_\_ \}.$$

[ii] The radius-r ***punctured ball*** is

$$\text{Ball}'_r(P) := \{ \_\_\_ \in \Omega \mid \_\_\_\_\_\_\_\_\_\_\_\_ \}.$$

[iii] For a seq. $\vec{\mathbf{b}} \subset \Omega$, the stmt $\lim(\vec{\mathbf{b}}) = P$ means .... For a fnc $f{:}\Omega\to\mathbb{R}$, the stmt $\lim_{Q\to P} f(Q) = 7$ means ....

[iv] A subset $C \subset \Omega$ is Ω-***closed*** if ....
A subset $K \subset \Omega$ is ***sequentially compact*** if ....
An ***open cover*** of a set $E \subset \Omega$ is .... A subset $E \subset \Omega$ is ***compact*** if ....

C6: Fix a compact MS (Ω, m).

[a] Prove that each closed subset $E \subset \Omega$ is compact. [*Hint:* The complement, $\Omega \smallsetminus E$, is Ω-open. Use it, together with a given open-cover of E, to produce an open-cover of Ω. Etc.]

[b] Prove that Ω is sequentially-compact. [*Hint:* FTSOC, consider a seq. $\vec{\mathbf{s}} \subset \Omega$ with no convergent subseq. Argue, WLOG, that $\vec{\mathbf{s}}$ consists of distinct pts. Now argue that $U := \Omega \smallsetminus \{s_n\}_{n=1}^{\infty}$ is open. Now...]

C7: State the Fundamental Thm of Arithmetic (about factoring into primes).
Use the FTArithmetic to give a careful proof that there are *no* posints n, d for which $\left[\frac{n}{d}\right]^5 = 2$.

C8: Define: "A number $\beta \in \mathbb{R}$ is ***"algebraic*** of degree-2006**"** IFF...". You may use "poly(nomial)" without defn, but if you use terms "ratpoly" or "intpoly" then you must define them. Note that ***zip*** is my name for the poly all of whose coeffs are zero.
Define: "A number $\tau \in \mathbb{R}$ is ***transcendental*** IFF..."

End of Prac-C

Print name .... Ord:

HONOR CODE: *"I have neither requested nor received help on this exam other than from my professor."*

Signature: ....