Binomial Coefficients

Jonathan L.F. King
University of Florida, Gainesville FL 32611-2082, USA
squash@ufl.edu
Webpage http://squash.1gainesville.com/
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Entrance. (Use 'ineq.' for "inequality" and 'coeff' for "coefficient". Use $\asymp$ for "asymptotic to"; $f(n) \asymp g(n)$ means that $\frac{f(n)}{g(n)} \to 1$ as $n \nearrow \infty$.)

Stirling's formula says that $n! \asymp \sqrt{2\pi n} \cdot [n/e]^n$. This implies that the central binomial coeff $\binom{2n}{n}$ is asymptotic to $4^n \cdot \frac{\text{Const}}{\sqrt{n}}$, where

$$\text{Const} \;=\; 1/\sqrt{\pi} \;\approx\; 0.564\,.$$

Overview. Fix a posint K for the rest of this note. Induction on $n \in [K \mathinner{..} \infty)$ will give

$$\alpha_n\textbf{:} \qquad \binom{2n}{n} \;\geq\; 4^n \cdot \frac{\mathbf{A}}{\sqrt{n}}$$

$$\beta_n\textbf{:} \qquad \binom{2n}{n} \;\leq\; 4^n \cdot \frac{\mathbf{B}}{\sqrt{T+n}}\,,$$

for all $n \in [K \mathinner{..} \infty)$, once non-negative constants $\mathbf{B}$,$\mathbf{A}$,T are chosen according to the following lemma. Necessarily, constants $\mathbf{B}$ and $\mathbf{A}$ will have to satisfy $\mathbf{B} \geq \frac{1}{\sqrt{\pi}} \geq \mathbf{A}$.

1.1: Central-coefficient lemma. *Define A_K st. $\text{LhS}(\alpha_K)$ equals $\text{RhS}(\alpha_K)$, i.e, let*

$$\textbf{1.2:} \qquad \mathbf{A} \;:=\; A_K \;:=\; \frac{\binom{2K}{K}}{4^K} \cdot \sqrt{K}\,.$$

For each $n \in [K \mathinner{..} \infty)$, then, inequality (α_n) holds.

Fix a real $T \geq \frac{K}{4K-1}$, then take $\mathbf{B}$ so as to give equality in (β_K). I.e, define

$$\textbf{1.3:} \qquad \mathbf{B} \;:=\; B_K \;:=\; \frac{\binom{2K}{K}}{4^K} \cdot \sqrt{T+K}\,.$$

Then inequality (β_n) holds for each $n \in [K \mathinner{..} \infty)$.

Lastly,

$$\gamma\textbf{:} \qquad \binom{2n}{n} \;\geq\; \frac{1}{2n} \cdot 4^n\,,$$

for each $n \in \mathbb{Z}_+$. ♢

Preliminaries. Let $\widehat{n}$ denote some unknown *positive* function of n, and let $\boxed{n} := 1/\widehat{n}$. Each of the three bounds above is OTForm $\widehat{n} \cdot 4^n$. Thus each inequality has the form

$$\dagger(n)\textbf{:} \qquad \binom{2n}{n} \;\gtrless\; 4^n \cdot \widehat{n}\,,$$

where relation $\gtrless$ is either "$\geq$" or "$\leq$".

Assume that we have verified $(\dagger)$ for some base-case value of n. The induction step is $\dagger(n{-}1) \implies \dagger(n)$. Here is the algebra:

$$\binom{2n}{n} = \binom{2n{-}2}{n{-}1} \cdot \frac{[2n-1]\cdot 2n}{n \cdot n} = 4 \cdot \binom{2n{-}2}{n{-}1} \cdot \frac{[2n-1]}{2n}$$

$$\textit{(by induction)} \quad \gtrless 4 \cdot \Big[4^{n-1} \cdot \widehat{n{-}1}\Big] \cdot \frac{[2n-1]}{2n}$$

$$= 4^n \cdot \widehat{n{-}1} \cdot \frac{[2n-1]}{2n}\,.$$

We want the RhS to $\gtrless 4^n \cdot \widehat{n}$, in order to continue the induction. So we wish to establish

$$\widehat{n{-}1} \cdot \frac{2n-1}{2n} \;\overset{?}{\gtrless}\; \widehat{n}\,.$$

Replace each $\widehat{n}$ by $1/\boxed{n}$. This rewrites our goal as

$$\ddagger\textbf{:} \qquad \frac{2n-1}{2n} \;\overset{?}{\gtrless}\; \frac{\boxed{n{-}1}}{\boxed{n}}\,. \qquad \square$$

Proof of (γ). Here, $\boxed{n} := 2n$, so the desired inequality $(\ddagger)$ is

$$\frac{2n{-}1}{2n} \;\overset{?}{\geq}\; \frac{2[n{-}1]}{2n}\,,$$

which is immediate. Finally, (γ) holds at $n{=}1$. ♦

Proof of (α). Let $\boxed{n} := \sqrt{n}$; the multiplicative constant $\mathbf{A}$ is irrelevant for $(\ddagger)$, the induction step. We wish to show that $\frac{2n-1}{2n} \geq \frac{\sqrt{n-1}}{\sqrt{n}}$. Squaring each side gives this equivalent ineq.

$$\Big[1 - \frac{1/2}{n}\Big]^2 \;\overset{?}{\geq}\; 1 - \frac{1}{n}\,.$$

But LhS^2 equals $\text{RhS} + [\frac{1/2}{n}]^2$, so the inequality holds. ♦

Proof of (β). Set $\boxed{n} := \sqrt{T+n}$.

Inequality (‡) becomes $\frac{2n-1}{2n} \le \frac{\sqrt{T+n-1}}{\sqrt{T+n}}$. This is equivalent $\big($since $n \ge 1$, so $T+n-1 \ge 0\big)$ to its square,

$$\left[1 - \frac{1}{2n}\right]^2 \quad \le \quad 1 - \frac{1}{T+n}\,.$$

Squaring-out the LhS leads to $\frac{1}{4n^2} - \frac{1}{n} \le -\frac{1}{T+n}$. So $\frac{4n-1}{4n^2} \ge \frac{1}{T+n}$, i.e $T+n \ge \frac{4n^2}{4n-1}$. Thus $T \ge \frac{1}{4} \cdot \frac{4n}{4n-1}$. And since the RhS of this is decreasing, each

1.4: $$T \quad \ge \quad \frac{K}{4K-1}$$

suffices for the induction to hold. ♦

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