Differentiating a bilinear function

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(Below, use ***VS*** for *vector space*, and ***IPS*** for *inner-product space*.)

Prolegomenon. In this pamphlet, all VSes are *real* VSes, $\mathbb{R}$-VSes.

The Product Rule from calculus states:

1a: *Suppose* $f, g : \mathbb{R} \to \mathbb{R}$ *are differentiable. Then so is their product, and*
$$[f \cdot g]' = [f \cdot g'] + [f' \cdot g] .$$

It turns out this generalizes.

1b: *For an* $\mathbb{R}$*-IPS* $\mathbf{U}$*: Suppose* $f, g : \mathbb{R} \to \mathbf{U}$ *are diff'able fncs. Then so is* $\langle f, g \rangle$*, and*
$$\langle f, g \rangle' = \langle f, g' \rangle + \langle f', g \rangle .$$

Also, for Physics problems in 3-dim'al Euclidean space:

1c: *Suppose* $f, g : \mathbb{R} \to \mathbb{R}^3$ *are diff'able. Then so are inner-product* ("dot-product") $\langle f, g \rangle$ *and cross-product. They satisfy*
$$\langle f, g \rangle' = \langle f, g' \rangle + \langle f', g \rangle , \quad \textit{and}$$
$$f \times g' = f \times g' + f' \times g .$$

All these raise the question (*not* "Beg the question", which means something different): What does it *mean* for a fnc $\mathbb{R} \to \mathbf{U}$ to be "differentiable"?

We suppose that $(\mathbf{U}, \|\cdot\|)$ is a *normed VS*. For a fnc $f : \mathbb{R} \to \mathbf{U}$ at a point $\boldsymbol{\tau} \in \mathbb{R}$, we can make sense of the *difference-quotient*

1d: $$\frac{f(\boldsymbol{\tau} + h) - f(\boldsymbol{\tau})}{h} , \quad \text{for non-zero } h \in \mathbb{R}.$$

Sending $h \to 0$ might give a $\|\cdot\|$-limit; if so, we denote the limit by $f'(\boldsymbol{\tau})$.

Consider normed VSes $\mathbf{U}, \mathbf{V}, \mathbf{X}$, a fnc $\Omega : \mathbf{U} \times \mathbf{V} \to \mathbf{X}$, and a point $P := (\mathbf{u}, \mathbf{v})$ in $\mathbf{U} \times \mathbf{V}$. Then Ω is "(jointly) continuous at P" if:

1e: *For all sequences* $\mathbf{a}_n \to \mathbf{u}$ *in* $\mathbf{U}$*, and* $\mathbf{b}_n \to \mathbf{v}$ *in* $\mathbf{V}$*, sequence*
$$\Omega(\mathbf{a}_n, \mathbf{b}_n) \text{ tends to } \Omega(\mathbf{u}, \mathbf{v}) \text{ in } \mathbf{X}.$$

2: Product-rule Theorem. *Consider normed vector-spaces* $\mathbf{U}, \mathbf{V}, \mathbf{X}$ *and differentiable functions* $\alpha : \mathbb{R} \to \mathbf{U}$ *and* $\beta : \mathbb{R} \to \mathbf{V}$*. Suppose* $[\![\cdot, \cdot]\!]$ *is a bilinear map* $\mathbf{U} \times \mathbf{V} \to \mathbf{X}$*.*

If $[\![\cdot, \cdot]\!]$ *is* (*jointly*) *continuous, then*
$$f(t) := [\![\alpha(t), \beta(t)]\!]$$
is differentiable, and

$*$: $$[\![\alpha, \beta]\!]' = [\![\alpha, \beta']\!] + [\![\alpha', \beta]\!] . \quad \diamond$$

Pf. Fix $\boldsymbol{\tau} \in \mathbb{R}$ and take a non-zero h. Then $f(\boldsymbol{\tau} + h) - f(\boldsymbol{\tau})$ equals
$$[\![\alpha(\boldsymbol{\tau} + h), \beta(\boldsymbol{\tau} + h)]\!] - [\![\alpha(\boldsymbol{\tau} + h), \beta(\boldsymbol{\tau})]\!]$$
$$+ [\![\alpha(\boldsymbol{\tau} + h), \beta(\boldsymbol{\tau})]\!] - [\![\alpha(\boldsymbol{\tau}), \beta(\boldsymbol{\tau})]\!] .$$

Using linearity in each argument separately, difference $f(\boldsymbol{\tau} + h) - f(\boldsymbol{\tau})$ equals
$$\left[\!\!\left[\alpha(\boldsymbol{\tau} + h), \frac{\beta(\boldsymbol{\tau}+h) - \beta(\boldsymbol{\tau})}{h}\right]\!\!\right] + \left[\!\!\left[\frac{\alpha(\boldsymbol{\tau}+h) - \alpha(\boldsymbol{\tau})}{h}, \beta(\boldsymbol{\tau})\right]\!\!\right] .$$

Sending $h \to 0$ yields ($*$), courtesy the (joint) continuity. ♦

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