Advanced Calc
MAA4102 & MAA5104

Home-β

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Definitions. Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$. A set S is ***countable*** if *there exists* an injection $S \hookrightarrow \mathbb{N}$. [This is equivalent, once S is non-void, to asking that there exists a surjection $\mathbb{N} \twoheadrightarrow S$.(AC)]

A set S is ***denumerable*** if it is bijective with $\mathbb{N}$.

Say that a polynomial $p()$ **"is n-topped"**, if $\text{Deg}(p) < n$. These are 3-topped polynomials: $x^2 - 2x$; $x + \sqrt{7}$; 17; zip. However $x^3 + x$ is *not* 3-topped.

An ***intpoly*** is a polynomial whose coefficients are integers. A number $\alpha \in \mathbb{R}$ is ***algebraic*** if it is a zero of some *non-zip* intpoly p, i.e, $p(\alpha) = 0$. The ***(algebraic) degree of*** α is the smallest $D \geqslant 0$ for which α is the zero of some degree-D intpoly.

For free. Here is a bijection $\sigma: 2\mathbb{N} \to \mathbb{N}$, where "$2\mathbb{N}$" shall mean $\{0,1\} \times \mathbb{N}$: For each bit b and natural ℓ, let $\sigma\big((b, \ell)\big) := 2\ell$, if $b = 0$, and $\sigma\big((b, \ell)\big)$ is $2\ell + 1$, if $b = 1$. Let $\tau()$ denote the inverse function $\sigma^{-1}()$.

Useful nomenclature. For a fnc $f: X \to Y$ and point $z \in X$, let $\text{Comp}(f, z)$ be $f(z)$ —which is a value in Y.

Suppose now that f is a point in: *The set of maps from X_1 to the set of maps from X_2 to ... the set of maps from X_N to B.* Let

$$\text{GenComp}\big(f, (x_1, x_2, \ldots, x_N)\big) \text{ mean } f\Big(x_1\Big)(x_2)\ldots(x_N).$$

This is a value in B.

Advice. Carefully follow the writing style described on *The Checklist*. Print a copy of your essay(s) each day, so that you always have a paper copy. Each team member is responsible to understand everything that the team hands in.

β1: a Give an explicit formula and/or computer program, possibly using $\lfloor \cdot \rfloor$ or $\lceil \cdot \rceil$, for a *bijection*

$$\gamma: \mathbb{N} \times \mathbb{N} \to \mathbb{N},$$

as well as a formula for its inverse-function $\delta()$. [*Hint:* What is the sum of the first k integers?]

b Please prove the following helpful LEMMA: *If $A_1, A_2, \ldots$ are each countable (subsets of a set X), then their union $\bigcup_1^\infty A_n$ is countable.*

β2: Recall, for sets D and C, that "C^D" denotes the *set* of **all** maps $D \to C$.

I For each three sets B, C, D give an *explicit* bijection

$$\Phi: [B^C]^D \to B^{C \times D}.$$

Make sure that your Φ works for arbitrary sets; infinite, finite (even empty). Let $\Theta() := \Phi^{-1}()$.

II Use maps σ and τ, as well as the idea of part (I), to give an particular bijection $\Upsilon: 3^{\mathbb{N}} \times 3^{\mathbb{N}} \to 3^{\mathbb{N}}$.

β3: a Let $\mathbb{P}_D$ be the set of D-topped intpolys. Let $\varphi: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be a particular bijection. (You could easily build φ from γ, but don't bother.) *In terms of* φ, give an *explicit* map F_D which is a bijection

$$F_D: \mathbb{P}_D \to \mathbb{Z}, \quad \text{for } D = 1, 2, 3, \ldots$$

[*Hint:* Induction on D.]

b **Fact:** *For $n \geqslant 0$, each degree-n polynomial has at most n zeros.* Use this to prove that Y_n is *countable*, where Y_n is the set of zeros of *non-zip* n-topped intpolys. [*Hint:* If "intpoly" is replaced by "poly" then Y_n is uncountable.]

c Prove that the set of *algebraic numbers* is countable. Conclude that there exist transcendental numbers. (Are there only one or two, or lots?)

β4: Show that a *continuous* function is determined by its values on any dense subset of its domain. That is, suppose that D is a dense subset of $\mathbb{R}$, and that $g, h: \mathbb{R} \to \mathbb{R}$ are continuous fncs with $g|_D = h|_D$. (I.e, their restrictions are equal.) Prove that $g = h$.

β5:

i Let $\alpha: 2^{\mathbb{N}} \to \mathbb{R}$ be some bijection, and let $\beta() := \alpha^{-1}()$. In terms of $\alpha()$ and $\beta()$, give an explicit bijection $G: \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$. Please let $H := G^{-1}$. [*Hint:* What letter comes next in $\alpha, \beta, \ldots$? You may want to first make $\overline{G}: [2^{\mathbb{N}}]^{\mathbb{N}} \to 2^{\mathbb{N}}$, which is related to G.]

ii Motivation: The set $\mathbb{R}^{\mathbb{R}}$ of *all* functions from $\mathbb{R}$ to $\mathbb{R}$ has cardinality $2^{\mathbb{R}}$. That is, $\mathbb{R}^{\mathbb{R}}$ has the same cardinality as the powerset of $\mathbb{R}$.

Let $\mathbf{C}$ denote the set of *continuous* functions $\mathbb{R} \to \mathbb{R}$. Show that there exists an *injection* $\mathbf{C} \hookrightarrow 2^{\mathbb{N}}$ (equivalently, an injection into $\mathbb{R}$). You will want to use part (i) and (β4). [*Hint:* Recall that $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$.]