Topo 1: MTG4302 *and* MTG5316 | **Home-B** | Prof. JLF King 6Sep2015

Info. Take-home B is due at the beginning of class on **Monday, 8 Dec., 1997**.

B1: Suppose $(S, \sqsubset)$ is a self-dense total order with no extrema and suppose that S is denumerable. Show that S is order-isomorphic with the set, $\mathbb{D}$, of dyadic rationals in $(0,1)$. [*Hint:* Divide and conquer.] Show that $\mathbb{Q}$ is order-isomorphic with $\mathbb{D}$, the set of of dyadic rationals in $(0,1)$.

B2: Suppose $\mathcal{T}$ and $\mathcal{M}$ are topologies on X. Say that $\mathcal{T}$ is ***sequentially-over*** $\mathcal{M}$, written $\mathcal{T} \succ \mathcal{M}$, if for every sequence $\vec{\mathbf{x}}$ and point z in X,

$$x_n \xrightarrow[n\to\infty]{\mathcal{T}} z \quad\implies\quad x_n \xrightarrow[n\to\infty]{\mathcal{M}} z$$

Suppose $\mathcal{T}$ is *locally countably generated*. Prove that $\mathcal{T} \succ \mathcal{M}$ implies $\mathcal{T} \supset \mathcal{M}$. [*Hint:* You need to show, for each $\mathcal{M}$-open set R and point $p \in R$, that there is an $\mathcal{T}$-open set $V \ni p$ with $V \subset R$. Prove this by contradiction: Consider $(U_n)_{n=1}^{\infty}$, a countable local $\mathcal{T}$-base at p , and assume that each U_n "sticks out" of R. Use this to produce a sequence of points which, although they $\mathcal{T}$-converge to p, *fail* to $\mathcal{M}$-converge to p. Now *carefully explain* why this argument proves the theorem.]

B3: From a metric space (X, m), construct the metric space $\Omega := \mathbf{C}_{\text{Bnd}}(X, \mathbb{R})$ with the supremum metric

$$d(f,g) \;:=\; \|f-g\|_{\sup}\,.$$

(Notes, P. 19; the set of *continuous* and *bounded* fncs.)

Prove that Ω is complete by first showing, for each d-Cauchy sequence $(f_n)_{n=1}^{\infty}$, that for all x the limit

$$h(x) \;:=\; \lim_{n\to\infty} f_n(x)$$

exists in $\mathbb{R}$, by using the completeness of $\mathbb{R}$. Next show that h is continuous; don't just cite uniform-convergence –give a *proof.* Finally, show that h is bnded and that $d(f_n, h) \to 0$.

B4: a (X, d) is a separable metric space and $E \subset X$ is a subset. Show that (E, d) is a separable. (So, in a MS, separability is hereditary. Note: This does not hold for general topological spaces.)

b Let $(\Omega, \mathcal{M})$ be the "tangent disk" topological space of §C, example 3.5, on P. 23 of the notes. Prove that Ω is *not* metrizable, as follows: Show that $\mathcal{M}$ is separable. Show that the relative topology induced on the horizontal-axis is not separable.

B5: We define two topologies, $\mathcal{D}$ and $\mathcal{H}$ on $\mathbb{R}$. Let $\mathcal{H}$ be generated by all half-open intervals $[a,b)$ with $a,b \in \mathbb{R}$. In contrast, let $\mathcal{D}$ be generated by the half-open intervals $[a,b)$ with *dyadic* endpoints, i.e, each of a and b is of the form $p/2^n$ where p and n are integers.

a Show that each topology is totally-disconnected.

b Show that $\mathcal{D}$ *is* metrizable. One approach is to note that, for each half-open interval such as $[0,1)$, the $\mathcal{D}$-topology makes this interval look like the standard Cantor set (divide and conquer) minus some points, and, of course, the Cantor set is metrizable. When all the dust has settled, you can give an *explicit* metric on $\mathbb{R}$ which induces $\mathcal{D}$.

c Show that $\mathcal{H}$ is LCG (easy), *but* is not metrizable. There are several ways to do this; here is one: Show that $\mathcal{H}$ is separable. Show that $\mathcal{H}$ is not countably generated. Now prove that every separable metric space is countably generated.

Extra Credit: Find a completely different proof that $\mathcal{H}$ is not metrizable.

End of Home-B