Probability II MAP6473 4810 **Solns-B** Prof. JLF King 10Sep2015

Tools. For each "speed" $s \in \mathbb{R}$, use E_s as a name for the map $\boxed{z \mapsto e^{isz}}$ on $\mathbb{R}$. Note that

1: $|e^{i\theta} - 1| \leqslant |\theta|$, for all real θ. This since chord⩽arc.

By the way, unmarked integrals $\int$ shall mean $\int_{\mathbb{R}}$.

2: Equi-cts Lemma. *On compact metric space J we have real-valued functions $h_n \to g$ pointwise. The convergence will be uniform, if $\{h_n\}_n$ is a uniformly equi-continuous family.* ◊

Proof. The limit g automatically has the same ε,δ-relation as the family; so we may replace each h_n by $h_n - g$. I.e, WLOGenerality

The $(h_n)_n$ *converge pointwise to zero.*

Given ε, take the corresponding δ from the family. Now pick a δ-dense set, $\mathcal{F}$, of points in J; we can take $\mathcal{F}$ **finite**, since J is compact.

Discarding the first few h_n, we now have

$$\forall n,\ \forall t \in \mathcal{F}: \quad |h_n(t)| \leqslant \varepsilon;$$

this, by the convergence to zero. For an arbitrary point $s \in J$ there is a point $t \in \mathcal{F}$ which is δ-close to s. Their h_n-values are thus ε-close. The upshot is that for all n: $|h_n(s)| \leqslant 2\varepsilon$.

Consequently, back in our original notation we have that $\|h_n - g\|_{\sup} \leqslant 2\varepsilon$, for all large n. ♦

3: Bnd Lemma. *On prob.space (Ω,μ), meas.map $f:\Omega\to\mathbb{R}$ has $f() \leqslant 7$. If $\int f \,d\mu \geqslant 7$ then $f() \overset{\text{a.e}}{=} 7$.* ◊

Proof. As $\{f < 7\} = \bigcup_j \{f \leqslant 7 - \frac{1}{j}\}$, I need but fix a number $L<7$ and show that $v := \mu(f() \leqslant L)$ is zero. But $\int f \leqslant [1-v]\cdot 7 + v \cdot L = 7 - v[7-L]$. So $v = 0$. ♦

B1: For a prob.meas. μ, let $\tilde{\mu}$ be "μ flipped"; so $\tilde{\mu}(B) := \mu(-B)$. Write the char.fnc $\Phi_{\tilde{\mu}}$ ITOf (In Terms Of) Φ_μ.

Let $\langle\mu;7\rangle$ and $\langle\mu;7,3\rangle$ be a translation and a translation-scaling of μ:

$$\langle\mu;7\rangle(B) := \mu(B-7);$$
$$\langle\mu;7,3\rangle(B) := \mu(3B-7).$$

Describe the char.fncs $\Phi_{\langle\mu;7\rangle}$ and $\Phi_{\langle\mu;7,3\rangle}$ ITOf Φ_μ.

Soln-**B1:** Let Φ mean Φ_μ.

The problem discusses affine maps of $\mathbb{R}$. Lets broaden our view to a general measurable map $Q:\mathbb{R}\circlearrowleft$ and define the "push forward measure" $\langle\mu;Q\rangle(B) := \mu\big(Q^{-1}(B)\big)$. Written with the indicator fnc, $\int \mathbf{1}_B \, d\langle\mu;Q\rangle = \int \mathbf{1}_B \circ Q \, d\mu$. So

4: $$\int f \, d\langle\mu;Q\rangle = \int f\circ Q \, d\mu,$$

for each measurable[♡1] fnc $f:\mathbb{R}\to\mathbb{C}$. Applied when $f := E_t$,

5: $$\Phi_{\langle\mu;Q\rangle}(t) = \int E_t \circ Q \, d\mu = \int e^{it\cdot Q(x)} \, d\mu(x).$$

Applying this with the *flip*, $Q(x) := -x$, gives

C_{Flip}: $\Phi_{\tilde{\mu}}(t) = \Phi(-t)$, i.e $\Phi_{\tilde{\mu}} = \overline{\Phi}$.

Here, ***overbar*** is complex conjugation.

The *translation map*, $Q(x) := x + 7$, produces

C_{Tr}: $\Phi_{\langle\mu;7\rangle}(t) = e^{it\cdot 7}\cdot\Phi(t)$, i.e $\Phi_{\langle\mu;7\rangle} = E_7\cdot\Phi$.

Affine map $Q(x) := \frac{x+7}{3}$ hands us

C_{Aff}: $\Phi_{\langle\mu;7,3\rangle}(t) = e^{it\cdot[7/3]}\cdot\Phi(t/3)$.

Soln

B2: Please do Billingley:**26.1** P.353. Use A,B for a,b. The "lattice" is $L := A + B\mathbb{Z}$, a scaled translation of the integers. For each integer n, there is a mass $m_n := \mathrm{P}(\mathsf{X} = A+Bn)$; these masses sum to 1.

Soln-**B2:** a With $\Phi := \Phi_{\mathsf{X}}$, and m_n the mass at point $A + nB$, we have that $\Phi(t)$ equals

$$\sum_{n\in\mathbb{Z}} m_n \cdot e^{it[A+Bn]} \overset{|\cdot|}{=\!=} \sum_n m_n \cdot e^{it\cdot Bn}.$$

At the speed $t := \frac{2\pi}{B}$ (which is non-zero) we have, for each integer n, that $e^{it\cdot Bn} = e^{i2\pi\cdot n} = 1$. The upshot is that $\boxed{\Phi(\frac{2\pi}{B}) = \exp(i2\pi\frac{A}{B})}$.

b Use $\mathbb{S}$ for the unit circle in $\mathbb{C}$. Fixing pos-real B, a "***B-lattice***" is a translate of $B\mathbb{Z}$. Our goal:

¥: *When μ a prob.meas: $\Phi_\mu(B) \in \mathbb{S}$ IFF μ is supported on some B-lattice.*

[♡1] An f is $\langle\mu;Q\rangle$-integrable iff $f\circ Q$ is μ-integrable.

Proof. The complex number $\Phi_\mu(B)$ is the integral $\boxed{\int_{\mathbb{S}} z\,\mathrm{d}\nu(z)}$ on the unit circle, against the *push-forward* measure $\boxed{\nu := \mu \circ E_B^{-1}}$. Our goal has transmogrified to showing ν a point-mass.

Translating μ on $\mathbb{R}$ corresponds to rotating[♡2] ν. So WLOG $\int_{\mathbb{S}} z\,\mathrm{d}\nu(z)$ is 1. Taking the real part, then, $\int_{\mathbb{S}} f(z)\,\mathrm{d}\nu(z) = 1$ where $f(z) := \mathrm{Re}(z)$.

Since $f \leqslant 1$, the Bnd Lemma (**??**) asserts $f \equiv 1$ ν-a.e. Hence ν is supported on $1 \in \mathbb{S}$, since all other $z \in \mathbb{S}$ have smaller real-part. ♦

C We have non-zero speeds $B\,\&\,C$ with irrational ratio $R := \frac{C}{B}$, such that $\Phi(B), \Phi(C) \in \mathbb{S}$. So our μ is supported (Thank (¥)ou!) on the intersection of *some* B-lattice with *some* C-lattice. Let z be a common point with positive μ-mass. Move μ by an affine map carrying z to zero, and moving the B-lattice to $\mathbb{Z}$. So the C-lattice has been carried to $R\mathbb{Z}$. And the only multiple of irrational R which is integral, is zero times R.

Soln

B3: Billingley:**26.2** P.353.

Soln-**B3:** Symmetry of μ means $\tilde{\mu} = \mu$ which, from (**??**), means that $\Phi_\mu = \overline{\Phi_\mu}$; thus Φ_μ is real.

The converse assumes that $\Phi_\mu = \overline{\Phi_\mu}$ which, again (**??**), means that $\Phi_{\tilde{\mu}} = \Phi_\mu$. The (non-trivial) uniqueness **thm 26.2** (P.346) forces $\tilde{\mu} = \mu$.

Soln

B4: Billingley:**26.5** P.354.

Soln-**B4:** An integrable h:ℝ↻ defines a signed-measure; use "Φ_h" for its characteristic fnc. Note that $\Phi_h(1)$ equals $\int_{\mathbb{R}} h$; this may be zero or negative. We strengthen the problem to:

6: Theorem. *On $\mathbb{R}$, suppose that $f, f', f'', \ldots, f^{(N)}$ are integrable (w.r.t Lebesgue measure). Then*

$$\Phi_f(t) \;=\; o(1/t^N)\,, \quad \text{as } N \to \infty.$$

[♡2] Map E_B is a group homomorphism. Hence it carries group translations to group translations.

Proof. Applying (**??**∗) repeatedly gives

$$\textbf{??}*: \qquad \Phi_f(t) \;=\; \left[-1/\boldsymbol{i}t\right]^N \cdot \Upsilon(t)\,,$$

where $\Upsilon := \Phi_{f^{(N)}}$ is the char.fnc of the signed-measure coming from $f^{(N)}$. This measure is abs.continuous, so our "Good" thm from class shows that $\Upsilon(t) \to 0$, as $t \to \infty$. ♦

7: Lemma. *Suppose fncs $h\,\&\,h'$ are Lebesgue integrable. Then $h(x) \to 0$ as $x \to \pm\infty$.* ♢

Proof. Since $h \in \mathbb{L}^1(\mathrm{Leb})$, automatically $\liminf_{z \to +\infty} |h(z)|$ is zero. Fixing ε, then, there are arbitrarily large pts z_0 so that $\boxed{|h(z_0)| < \varepsilon}$. Pick one so large that, estimating the derivative,

$$\int_{z_0}^{\infty} |h'| \;\leqslant\; \varepsilon\,.$$

For an arbitrary $x > z_0$, the FTCalculus gives

$$h(x) \;=\; h(z_0) + \int_{z_0}^{x} h'\,.$$

Thus $|h(x)| \;=\; \varepsilon + \int_{z_0}^{x} |h'| \;\leqslant\; 2\varepsilon$. ♦

A corollary of (**??**) is

$$\textbf{??}*: \qquad \forall t \in \mathbb{R}: \quad \mathrm{e}^{\boldsymbol{i}tx} h(x)\Big|_{x=-\infty}^{x=+\infty} \;=\; 0\,,$$

since each $x \mapsto \mathrm{e}^{\boldsymbol{i}tx}$ is bounded.

8: Prop'n. *Suppose that $h\,\&\,h'$ are (Lebesgue) integrable. Then* $\boxed{\textit{for each real } t \neq 0}$:

$$\textbf{??}*: \qquad \Phi_h(t) \;=\; [-1/\boldsymbol{i}t] \cdot \Phi_{h'}(t)\,.$$

Proof. Integrating by-parts, $\Phi_h(t)$ equals

$$\int \mathrm{e}^{\boldsymbol{i}tx} h(x)\,\mathrm{d}x \;=\; V \;-\; \int \frac{1}{\boldsymbol{i}t} \cdot \mathrm{e}^{\boldsymbol{i}tx} \cdot h'(x)\,\mathrm{d}x\,,$$

where $V := \frac{1}{\boldsymbol{i}t} \mathrm{e}^{\boldsymbol{i}tx} h(x)\Big|_{x=-\infty}^{x=+\infty}$. And V is zero, courtesy (**??**∗). ♦

Soln

B5: Bill:**26.15** P.355. Remember the tool of partitioning a large compact interval into short subintervals, where all the separation points are continuity-points of μ.

Soln-**B5:** Rename the μ_n to ν_n.

a Fix ε. Let $\delta := \varepsilon/L$, where L is taken (courtesy tightness) large enough that

9: $$\forall \nu : \quad \nu(\mathbb{R} \setminus J) \leqslant \varepsilon ,$$

on interval $J := [-L, L]$. Algebraically, $\mathsf{e}^{sx} - \mathsf{e}^{tx}$ equals $\mathsf{e}^{tx}[\mathsf{e}^{[s-t]x} - 1]$. For s and t real, use (**??**) for

$$|\mathsf{e}^{sx} - \mathsf{e}^{tx}| \leqslant \Big|[s-t] \cdot x\Big| .$$

So for $|s-t| \leqslant \delta$ and $x \in J$, we conclude that

$$|\mathsf{e}^{sx} - \mathsf{e}^{tx}| \leqslant \delta \cdot |x| \leqslant \varepsilon .$$

Employ (**??**), then fact $\mu(J) \leqslant 1$, to obtain

$$\begin{aligned} |\Phi_\nu(s) - \Phi_\nu(t)| &\leqslant 2\varepsilon + \int_J |\mathsf{e}^{sx} - \mathsf{e}^{tx}| \cdot \mathrm{d}\nu(x) \\ &\leqslant 2\varepsilon + \int_J \varepsilon \cdot \mathrm{d}\nu(x) = 3\varepsilon . \end{aligned}$$

Thus $\{\Phi_{\nu_n}\}_n$ is a uniformly equi-cts family of fncs.

b A bnded set in $\mathbb{R}$ is included in some compact interval, J. Equi-cts Lemma **??**, with $h_n := \Phi_{\nu_n}$ and $g := \Phi_\mu$, gives uniform convergence.

c In distribution, $\boldsymbol{\delta}_{1/n} \to \boldsymbol{\delta}_0$. For every real t,

$$|\Phi_{\boldsymbol{\delta}_0}(t) - \Phi_{\boldsymbol{\delta}_{1/n}}(t)| = |1 - \mathsf{e}^{it/n}| .$$

At $t := \boldsymbol{\pi} n$, the distance is 2. This implies that $\|\Phi_{\boldsymbol{\delta}_0} - \Phi_{\boldsymbol{\delta}_{1/n}}\|_{\sup} = 2$, for each posint n.

Soln

B6: Bill:**24.6** P.326. As it was stated in class, the pointwise Ergodic Theorem applies to $\mathbb{L}^1$-fncs.

Soln-**B6:** We generalize, removing the non-negativity condition and asking merely that

$$\mathsf{E}(f^+) = \infty \quad \text{and} \quad \mathsf{E}(f^-) < \infty .$$

Given a posint K we can pick a large integer $\widehat{K}$ so that the "cut-off fnc"

$$g_K() := \operatorname{Min}\Big(\widehat{K},\ f_K()\Big) \quad \text{has} \quad \int g_K \,\mathrm{d}\mu \geqslant K .$$

Off of a nullset $\mathcal{N}_K$ we have pointwise convergence of the Cesàro averages:

$$\mathbb{A}_N(g_K) \longrightarrow \int g_K \text{, as } N \to \infty.$$

Now $\mathcal{M} := \cup_K \mathcal{N}_K$ is a nullset. Each $\mathbb{A}_N$ is order-preserving and so, at a point $\omega \in \Omega \setminus \mathcal{M}$,

$$\begin{aligned} \liminf_{N\to\infty} \mathbb{A}_N\big(f\big)(\omega) &\geqslant \operatorname{liminf}_N \mathbb{A}_N\big(g_K\big)(\omega) \\ &= \int g_K \geqslant K . \end{aligned}$$

Now take a supremum over K. ♦

Soln

That was fun! Let's do this Again!