Miscellaneous Algebra facts

J.L.F. King

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Ways to count in groups

The symbol $G \circlearrowleft \Omega$ means that gp G ***acts on*** set Ω; there is a gp-hom $\boxed{\psi\colon G \to \mathbb{S}_\Omega}$. For $h \in G$ and $\omega \in \Omega$, write the gp-action as $\psi_h(\omega)$ or $h(\omega)$ or just $h\omega$. Define the ***orbit*** and ***stabilizer*** of a point ω, and the ***fixed-pt set*** of a group-element h:

$$\begin{aligned}
\mathcal{O}(\omega) \;&=\; \mathcal{O}_G(\omega) \;\coloneqq\; \{h\omega \mid h \in G\} && \subset\; \Omega\,;\\
\mathrm{Stab}(\omega) \;&=\; \mathrm{Stab}_G(\omega) \;\coloneqq\; \{h \in G \mid h(\omega) = \omega\} && \subset\; G\,;\\
\mathrm{Fix}(h) \;&=\; \mathrm{Fix}_G(h) \;\coloneqq\; \{\omega \in \Omega \mid h(\omega) = \omega\} && \subset\; \Omega\,;\\
\mathrm{Orbits}_\Omega \;&=\; \mathrm{Orbits}_{G \circlearrowleft \Omega} \;\coloneqq\; \{\text{Set of } G\text{-orbits}\} && \subset\; \mathcal{P}(\Omega)\,.
\end{aligned}$$

Subgp $\mathrm{Stab}(\omega)$ is a *normal-subgp of G* exactly when $|\mathcal{O}(\omega)| = 1$; i.e., rarely.

1: $$\forall f \in G\colon \qquad f \cdot \mathrm{Stab}(\omega) \cdot f^{-1} \;=\; \mathrm{Stab}(f(\omega))\,.$$

2: Orb-Stab Lemma. *For each $\omega \in \Omega$:*

∗: $$|\mathrm{Stab}(\omega)| \cdot |\mathcal{O}(\omega)| \;=\; |G|\,. \qquad \Diamond$$

Pf. Let $H \coloneqq \mathrm{Stab}(\omega)$. Two elts $g, f \in G$ are "*orbit-equivalent*", $g \sim f$, if $g(\omega) = f(\omega)$. Evidently, the equiv-class of g is simply the left coset gH of the stabilizer of ω. These equiv-classes partition G, hence $(*)$. ♦

3: Burnside's Lemma. *Counting cardinalities,*

†: $$\sum_{\omega\in\Omega}|\mathrm{Stab}(\omega)| \overset{\#}{=} \Big\{(g,\omega) \;\Big|\; g(\omega)=\omega\Big\} \overset{\#}{=} \sum_{g\in G}|\mathrm{Fix}(g)|\,.$$

Counting the number of G-orbits, then,

‡: $$^{\#}\mathrm{Orbits}_{\Omega} \;=\; \frac{1}{|G|}\cdot\sum_{g\in G}|\mathrm{Fix}(g)| \;=\; \begin{bmatrix}\text{\# of tokens fixed by an}\\ \text{average element of } G\end{bmatrix}.$$ ◊

Proof. The # of G-orbits is $\sum_{\mathcal{O}\,\in\,\mathrm{Orbits}} 1 = \sum_{\mathcal{O}\in\mathrm{Orbits}}\sum_{\omega\in\mathcal{O}}\frac{1}{|\mathcal{O}(\omega)|} = \sum_{\omega\in\Omega}\frac{1}{|\mathcal{O}(\omega)|}$. Our Orb-Stab lemma says $\frac{1}{|\mathcal{O}(\omega)|} = \frac{|\mathrm{Stab}(\omega)|}{|G|}$. Hence

$$^{\#}\mathrm{Orbits}_{\Omega} \;=\; \frac{1}{|G|}\cdot\sum_{\omega\in\Omega}|\mathrm{Stab}(\omega)| \overset{(3\dagger)}{=\!=} \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)|\,.$$ ♦

Applications of counting. Color the 6-faces of a cube red, white and blue. How many distinct colorings are there, up to orientation-preserving congruence?

We will use Burnside's Lemma.

What isometry g?	How many such g?	$^{\#}\mathrm{Fix}(g) = 3^F$	$F := {}^{\#}$[Face-orbits under g]
Id	1	3^6	1+1+1+1+1+1
FaceRot 180°	$\frac{6}{2}\cdot 1 = 3$	3^4	1+2+2+1
FaceRot 90°	$\frac{6}{2}\cdot 2 = 6$	3^3	1+4+1
EdgeRot 180°	$\frac{12}{2}\cdot 1 = 6$	3^3	2+2+2
VertexRot 120°	$\frac{8}{2}\cdot 2 = 8$	3^2	3+3

The sum $\frac{1}{24}\cdot[1\cdot 3^6 + 3\cdot 3^4 + 6\cdot 3^3 + 6\cdot 3^3 + 8\cdot 3^2]$ equals 57.

Using κ many colors, $^{\#}\text{Face-colorings}(\kappa) = \frac{1}{24}\cdot[\kappa^6 + 3\kappa^4 + 12\kappa^3 + 8\kappa^2]$.

Cubing the cube

In one swell foop, let's count *face-colorings* [our previous computation], *vertex-colorings*, and *edge-colorings*. [As before, use κ for the number of colors.]

$\mathbb{S}_4$: Permutation of hyper-diags	Isometry type	Order	# group-elts of this type	$\mathbb{S}_6$: Face cycle-sig	$\mathbb{S}_8$: Vertex cycle-sig	$\mathbb{S}_{12}$: Edge cycle-sig
$\lceil 1^4 \rfloor$	*Id*	1	1	$\lceil 1^6 \rfloor$	$\lceil 1^8 \rfloor$	$\lceil 1^{12} \rfloor$
$\lceil 1^1, 3^1 \rfloor$	VertexRot 120°	3	$\frac{8}{2} \cdot 2 = 8$	$\lceil 3^2 \rfloor$	$\lceil 1^2, 3^2 \rfloor$	$\lceil 3^4 \rfloor$
$\lceil 2^2 \rfloor$	FaceRot 180°	2	$\frac{6}{2} \cdot 1 = 3$	$\lceil 1^2, 2^2 \rfloor$	$\lceil 2^4 \rfloor$	$\lceil 2^6 \rfloor$
$\lceil 4^1 \rfloor$	FaceRot 90°	4	$\frac{6}{2} \cdot 2 = 6$	$\lceil 1^2, 4^1 \rfloor$	$\lceil 4^2 \rfloor$	$\lceil 4^3 \rfloor$
$\lceil 1^2, 2^1 \rfloor$	EdgeRot 180°	2	$\frac{12}{2} \cdot 1 = 6$	$\lceil 2^3 \rfloor$	$\lceil 2^4 \rfloor$	$\lceil 1^2, 2^5 \rfloor$
$\lceil 1^4 \rfloor$	Antipodal map $x \mapsto -x$; i.e, point reflection.	2	1	$\lceil 2^3 \rfloor$	$\lceil 2^4 \rfloor$	$\lceil 2^6 \rfloor$
$\lceil 1^1, 3^1 \rfloor$	Vertex-Rotation by 120°, then antipodal map.	6	8	$\lceil 6^1 \rfloor$	$\lceil 2^1, 6^1 \rfloor$	$\lceil 6^2 \rfloor$
$\lceil 2^2 \rfloor$	Face-Rotation by 180°, then antipodal map.	2	3	$\lceil 1^4, 2^1 \rfloor$	$\lceil 2^4 \rfloor$	$\lceil 1^4, 2^4 \rfloor$
$\lceil 4^1 \rfloor$	Face-Rotation by 90°, then antipodal map.	4	6	$\lceil 2^1, 4^1 \rfloor$	$\lceil 4^2 \rfloor$	$\lceil 4^3 \rfloor$
$\lceil 1^2, 2^1 \rfloor$	Edge-Rotation by 180°, then antipodal map.	2	6	$\lceil 1^2, 2^2 \rfloor$	$\lceil 1^4, 2^2 \rfloor$	$\lceil 1^2, 2^5 \rfloor$

Alternate descriptions of some of the orientation-reversing maps:

Vertex-Rotation 120°, then antipodal map. = Reflection across equatorial plane through 6 edge-midpoints, then rotate 60°.

Face-Rotation by 180°, then antipodal map. = Reflection across the plane that passes through 4 edge-midpoints.

Edge-Rotation by 180°, then antipodal map. = Reflection across the plane passing through 4 vertices.

Number of colorings.

Positive orientations:

$$\#\mathrm{Face}(\kappa) = \tfrac{1}{24}\cdot[\kappa^6 + 3\kappa^4 + 12\kappa^3 + 8\kappa^2];$$

$$\#\mathrm{Vertex}(\kappa) = \tfrac{1}{24}\cdot[\kappa^8 + 17\kappa^4 + 6\kappa^2];$$

$$\#\mathrm{Edge}(\kappa) = \tfrac{1}{24}\cdot[\kappa^{12} + 6\kappa^7 + 3\kappa^6 + 8\kappa^4 + 6\kappa^3];$$

All orientations:

$$\#\mathrm{Face}_\pm(\kappa) = \tfrac{1}{48}\cdot[\kappa^6 + 3\kappa^5 + 9\kappa^4 + 13\kappa^3 + 14\kappa^2 + 8\kappa];$$

$$\#\mathrm{Vertex}_\pm(\kappa) = \tfrac{1}{48}\cdot[\kappa^8 + 6\kappa^6 + 21\kappa^4 + 20\kappa^2];$$

$$\#\mathrm{Edge}_\pm(\kappa) = \tfrac{1}{48}\cdot[\kappa^{12} + 3\kappa^8 + 12\kappa^7 + 4\kappa^6 + 8\kappa^4 + 12\kappa^3 + 8\kappa^2].$$

The *Energetic Reader* checks that these polynomials are $\mathbb{Z}$-valued on $\mathbb{Z}$. BTWay, coloring the four hyper-diagonals gives $\#\mathrm{HyperDiag}(\kappa) = \frac{1}{24}\cdot[\kappa^4 + 6\kappa^3 + 11\kappa^2 + 6\kappa]$.

Here are two challenge questions:

Q1: Let Γ, the orientation-preserving isometry group of the cube, act on the <u>edges</u> of the cube.

> *Under Γ-equivalence, must the number of 2-colorings of the edges equal the number of ways of putting an orientation* (an arrow) *on each edge?*

Q2: For $h \in \mathbb{S}_N$, let $\mathrm{Cyc}(h)$ count the # of (disjoint) cycles of h. [E.g, $h = (0,1,2)(3,4,5)(6)(7,8)(9)$ has $\mathrm{Cyc}(h) = 5$; here $N = 10$.] Prove, for $N = 1, 2, 3, \ldots$, that

$$\Big[\sum_{h \in \mathbb{S}_N} 2^{\mathrm{Cyc}(h)}\Big] \;=\; [N+1]!\,.$$