Affine maps of the plane

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Entrance. A matrix $\mathsf{M} := \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ acts on a point $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$, sending it to $\mathsf{M}\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$. Let $\widehat{\imath} := \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\widehat{\jmath} := \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. Let

$$\mathsf{R}_\theta := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

be the std rotation matrix.

Recall that SL_2, the ***special linear group***, is the group (sealed under matrix-mult and matrix-inverse) of 2×2 matrices M with $\mathrm{Det}(\mathsf{M}) = 1$. Each $\mathsf{M}\in\mathrm{SL}_2$ is ***OPAP***: ***O****rientation* ***P****reserving*, since $\mathrm{Det}(\mathsf{M}) > 0$; and ***A****rea* ***P****reserving*, since $|\mathrm{Det}(\mathsf{M})| = 1$.

Shears. An $m \in \mathbb{R}$ yields horizontal/vertical shears:

$$\mathsf{H}_m := \left[\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right], \qquad \mathsf{V}_m := \left[\begin{smallmatrix} 1 & 0 \\ m & 1 \end{smallmatrix}\right].$$

Abbreviate "horizontal(ly) shear" by ***hshear***, and "vertical(ly) shear" by ***vshear***. Call a *horizontal-or-vertical shear* a ***perp-shear***. Finally, let Σ denote the group generated by perp-shears.

1: Perp-shear Lemma. *Rotation* $\mathsf{R}_\pi = \mathsf{R}_{180°}$ *is a product of perp-shears:* $\mathsf{H}_{-2}\mathsf{V}_1\mathsf{H}_{-2}\mathsf{V}_1$.

a: *For each angle* θ*, rotation* R_θ *is a product of at most 5 perp-shears.*

b: *The group generated by the perp-shears is precisely* SL_2. ◊

Pf of (a). WLOG $\theta \in (0,\pi)$. Let $\mathbf{q} = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ be the *unit-vector* at angle θ. Vertically shear $\widehat{\imath}$ up to height y, then over to be $\mathbf{q}$, i.e take $\alpha \in \mathbb{R}$ st. $[\mathsf{H}_\alpha\mathsf{V}_y]\widehat{\imath} = \mathbf{q}$. This action moves $\widehat{\jmath}$ to some vector $\mathbf{w} := [\mathsf{H}_\alpha\mathsf{V}_y]\widehat{\jmath}$.

Let $\mathbb{L}$ be the line parallel to vector $\mathbf{q}$, and passing through point $\mathsf{R}_{\pi/2}(\mathbf{q})$. Since shears are OPAP, *this* $\mathbf{w}$ *must lie on* $\mathbb{L}$.

Take the $z \in \mathbb{R}$ which vshears $\mathbf{q}$ onto the horiz-axis, i.e $\mathsf{V}_z\,\mathbf{q} = \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]$. For each $\beta \in \mathbb{R}$, then, the hshear H_β (fixes $\mathsf{V}_z\mathbf{q}$). Since $\mathsf{V}_z\mathbf{q}$ is on the horiz-axis, point $\mathsf{V}_z\mathbf{w}$ can*not* be [they are two edges of an area=1 parallelogram]. Hence $\{[\mathsf{H}_\beta\mathsf{V}_z]\mathbf{w} \mid \beta \in \mathbb{R}\}$ is an entire line (horizontal, since vector $\mathsf{V}_z\mathbf{q}$ is horizontal). Letting $\mathsf{T}_\beta := \mathsf{V}_{-z}\mathsf{H}_\beta\mathsf{V}_z$, then,

$$\{[\mathsf{T}_\beta]\mathbf{w} \mid \beta \in \mathbb{R}\} \text{ is all of } \mathbb{L}.$$

Thus there is a particular β-value, b, st. $[\mathsf{T}_b]\mathbf{w}$ is orthogonal to $[\mathsf{T}_b]\mathbf{q} \overset{\text{note}}{=\!=} \mathbf{q}$. Since $\mathbf{q}$ has length 1, our $[\mathsf{T}_b]\mathbf{w}$ must have length 1. Thus $[\mathsf{T}_b]\mathbf{w}$ is $[\mathsf{T}_b]\mathbf{q}$ hit by $\mathsf{R}_{\pi/2}$. I.e,

$$[\mathsf{T}_b]\mathbf{q} = [\mathsf{R}_\theta]\widehat{\imath}, \quad \text{and}$$
$$[\mathsf{T}_b]\mathbf{w} = [\mathsf{R}_\theta]\widehat{\jmath}.$$

Consequently

$$[\mathsf{R}_\theta]\widehat{\imath} = [\mathsf{T}_b]\mathbf{q} = [\mathsf{T}_b\mathsf{H}_\alpha\mathsf{V}_y]\widehat{\imath}, \quad \text{and}$$
$$[\mathsf{R}_\theta]\widehat{\jmath} = [\mathsf{T}_b]\mathbf{w} = [\mathsf{T}_b\mathsf{H}_\alpha\mathsf{V}_y]\widehat{\jmath}.$$

Thus R_θ equals $\mathsf{T}_b\mathsf{H}_\alpha\mathsf{V}_y$, a product of 5 perp-shears. ♦

Proof of (b). To show that a $\mathsf{T} \in \mathrm{SL}_2$ is a perp-shear product, let $\mathbf{u} := \mathsf{T}\widehat{\imath}$ and $\mathbf{v} := \mathsf{T}\widehat{\jmath}$. We'll carry pair $(\mathbf{u},\mathbf{v})$ to $(\widehat{\imath},\widehat{\jmath})$ via perp-shears.

Take a rotation R st. $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] := \mathsf{R}\mathbf{u}$ has $0 < y < 1$. So there is an hshear H st. $\mathsf{HR}\mathbf{u}$ has length 1. Now take the rotation R' st. $\mathsf{R}'\mathsf{HR}\mathbf{u}$ is $\widehat{\imath}$.

All this has carried $\mathbf{v}$ to $\mathbf{v}' := [\mathsf{R}'\mathsf{HR}]\mathbf{v}$. Since $(\widehat{\imath},\mathbf{v}')$ defines a parallelogram with signed-area 1, there is a (unique) hshear H' st. $\mathsf{H}'\mathbf{v}' = \widehat{\jmath}$. The upshot: $\mathsf{H}'\mathsf{R}'\mathsf{HR}$ carries pair $(\mathbf{u},\mathbf{v})$ to $(\widehat{\imath},\widehat{\jmath})$. So $\mathsf{T} := [\mathsf{H}'\mathsf{R}'\mathsf{HR}]^{-1}$. ♦

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