Staple!

Team A___

Sets and Logic
MHF3202 7860

Home-A

Prof. JLF King
Wednesday, 31Jan2024

Due **BoC, Wedn, 07Feb2024**, wATMP! **Print this problem-sheet**; it is the first page of your write-up, with the blanks filled in (handwritten). Write **DNE** if the object does not exist or the operation cannot be performed. NB: **DNE** $\neq \{\} \neq 0$. [Put ordinal, Team-# and sign HONOR CODE.]

A1: *Show no work.*

___ a Compute the real $\alpha =$ .... such that

$$*: \qquad 3^{\alpha} \cdot \sum_{k=0}^{4000} \binom{4000}{k} 2^k \;=\; \sum_{j=0}^{1995} \binom{1995}{j} 8^j .$$

[*Hint:* The Binomial Theorem]

___ b $\forall x,z \in \mathbb{Z}$ *with* $x < z$, $\exists y \in \mathbb{Z}$ *st.:* $x < y < z$. T F

$\forall x,z \in \mathbb{Q}$ *with* $x \neq z$, $\exists y \in \mathbb{R}$ *st.:* $x < y < z$. T F

For all sets Ω, *there exists a fnc* $f{:}\mathbb{R}{\to}\Omega$. T F

___ c The number of ways of picking 42 objects from 70 types is $\left[\!\left[{70 \atop 42} \right]\!\right] \stackrel{\text{Binom}}{\underset{\text{coeff}}{=\!=}} \Big($ .... $\Big)$. And

$\left[\!\left[{70 \atop 42} \right]\!\right] = \left[\!\left[{T \atop K} \right]\!\right]$, where $T=$ .... $\neq 70$, and $K=$ ....

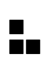
Carefully TYPE *your two essays, double–spaced. I suggest* LaTeX.

A2: Let $\mathbf{E}_n$ be the equilateral triangle with side-length 2^n. This $\mathbf{E}_n$ can be tiled in an obvious way by 4^n many little-triangles [copies of $\mathbf{E}_0$]; see picture on blackboard. The "***punctured*** $\mathbf{E}_n$", written $\widetilde{\mathbf{E}_n}$, has its topmost copy of $\mathbf{E}_0$ removed.

A ***(trape)zoid***, $\mathbf{T}$, comprises three copies of $\mathbf{E}_0$ glued together in a row, rightside-up, upside-down, rightside-up. [A ***zoid-tiling*** allows all six rotations of $\mathbf{T}$.]

i PROVE: *For each n, board $\widetilde{\mathbf{E}_n}$ admits a zoid-tiling.*

ii Let Δ_k be the equilateral triangle of sidelength k; so $\mathbf{E}_n$ is Δ_{2^n}. Triangle Δ_k comprises k^2 little-triangles.
For what values of k does Δ_k admit a zoid-tiling?
For which k does $\widetilde{\Delta_k}$ admit a zoid-tiling?

iii An ***Lmino*** (pron. "ell-mino") comprises three  squares in an "L" shape (all four orientations are allowed).

Let $\mathbf{S}_n$ be the $2^n{\times}2^n$ square board, comprising 4^n ***squaries*** (little squares). Have $\widetilde{\mathbf{S}_n}$ be the board with one corner squarie removed. Shown in class is an inductive proof that each $\widetilde{\mathbf{S}_n}$ is Lmino-tilable (by $[4^n - 1]/3$ Lminos, of course). Further, with $\mathbf{S}'_n$ denoting $\mathbf{S}_n$ with an *arbitrary* puncture, we proved that every $\mathbf{S}'_n$ is Lmino-tilable.

Generalize this to three-dimensions. Let $\mathbf{C}_n$ denote the $2^n{\times}2^n{\times}2^n$ cube, $\widetilde{\mathbf{C}_n}$ the corner-punctured cube, and let $\mathbf{C}'_n$ be $\mathbf{C}_n$ but with an arbitrary ***cubie*** removed.

What is the 3-dimensional analog of an Lmino? Calling it a "**3-*mino***", how many cubies does it have? [Provide a drawing of your 3-mino.] PROVE: *Every $\mathbf{C}'_n$ admits a 3-mino–tiling.* [Provide also pictures showing your ideas.]

iv Generalize to K-dim(ensional) space, with $\mathbf{C}_{n,K}$ being the $2^n{\times}\overset{K}{\cdots}{\times}2^n$ cube, having $[2^n]^K = 2^{nK}$ many K-dim'al cubies. As before, let $\mathbf{C}'_{n,K}$ be $\mathbf{C}_{n,K}$ with an arbitrary cubie removed.

*What is your **K-mino** with which you will tile, and how many cubies does it have?* (So a 2-mino is our Lmino.) PROVE: *Every $\mathbf{C}'_{n,K}$ admits a K-mino–tiling.*

A3: An integer-valued list $\mathcal{L} := (n_1, n_2, n_3, \ldots, n_9)$ is indexed by interval-of-integers $J := [1\,..\,9]$.

This J has .... *non-void* subsets.

And J has .... *non-void* subintervals. (Note: $[4\,..\,6]$ is a length-3 subinterval, and $[8\,..\,8]$ is a length-1 subinterval.)

α Use PHP [Pigeon-hole Principle] to prove for each $\mathcal{L}$ as above that: *There exists a non-void set* $\Omega \subset J$ *of indices, such that* $\left[\sum_{j\in\Omega} n_j\right] \bullet 9$. You may use $\equiv$ for $\equiv_9$ i.e, congruence mod-9.

β Produce an *interesting* generalization/companion of the above the PHP problem, then solve it.

A1:	___ ___	75pts
A2:	___ ___ ___	165pts
A3:	___ ___	80pts
Total:	___ ___ ___	320pts

HONOR CODE: *"I have neither requested nor received help on this exam other than from my team-mates and my professor (or his colleague)."* *Name/Signature/Ord*

Ord: ....

Ord: ....

Ord: ....