Staple!

Team A ______________________

Sets and Logic MHF3202 12A3 — **Home-A** — Prof. JLF King 12Oct2021

Due **BoC, Wednesday, 20Oct2021**, wATMP!
Write **DNE** if the object does not exist or the operation cannot be performed. NB: **DNE** $\neq \{\} \neq 0$.

A1: *Show no work.*

10 20 a) The ***Threeish-numbers*** comprise $\mathcal{T} := 1 + 3\mathbb{N}$.
$\mathcal{T}$-number $385 \overset{\text{note}}{=\!=} 35 \cdot 11$ is $\mathcal{T}$-*irreducible*: *T* *F*

Threeish $N := 85$ is **not** $\mathcal{T}$-*prime* because $\mathcal{T}$-numbers $J :=$ ________ and $K :=$ ________ satisfy that $N \bullet\!| [J \cdot K]$, **yet** $N \nmid J$ and $N \nmid K$.

Also, $\mathcal{T}$-GCD$(175, 70) =$ ________.

20 b) On $\Omega := [1\,..\,29] \times [1\,..\,29]$, define binary-relation **C** by: $(x, \alpha)\ \mathbf{C}\ (y, \beta)$ IFF $x \cdot \beta \equiv_{30} y \cdot \alpha$. Statement "*Relation* **C** *is an* ***equivalence relation***" is: *T* *F*

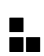
Carefully TYPE *your essays, double–spaced. I suggest* LaTeX, *but other systems are ok too.*

A2: Let $\mathbf{E}_n$ be the equilateral triangle with side-length 2^n. This $\mathbf{E}_n$ can be tiled in an obvious way by 4^n many little-triangles [copies of $\mathbf{E}_0$]; see picture on blackboard. The "***punctured*** $\mathbf{E}_n$", written $\widetilde{\mathbf{E}_n}$, has its topmost copy of $\mathbf{E}_0$ removed.

A ***(trape)zoid***, **T**, comprises three copies of $\mathbf{E}_0$ glued together in a row, rightside-up, upside-down, rightside-up. [A ***zoid-tiling*** allows all three rotations of **T**.]

i) PROVE: *For each n, board $\widetilde{\mathbf{E}_n}$ admits a zoid-tiling.*

ii) Let Δ_k be the equilateral triangle of sidelength k; so $\mathbf{E}_n$ is Δ_{2^n}. Triangle Δ_k comprises k^2 little-triangles.
For what values of k does Δ_k admit a zoid-tiling?
For which k does $\widetilde{\Delta_k}$ admit a zoid-tiling?

iii) An ***Lmino*** (pron. "ell-mino") comprises three  squares in an "L" shape (all four orientations are allowed).

Let $\mathbf{S}_n$ be the $2^n \times 2^n$ square board, comprising 4^n ***squaries*** (little squares). Have $\widetilde{\mathbf{S}_n}$ be the board with one corner squarie removed. Shown in class is an inductive proof that each $\widetilde{\mathbf{S}_n}$ is Lmino-tilable (by $[4^n - 1]/3$ Lminos, of course). Further, with $\mathbf{S}'_n$ denoting $\mathbf{S}_n$ with an *arbitrary* puncture, we proved that every $\mathbf{S}'_n$ is Lmino-tilable.

Generalize this to three-dimensions. Let $\mathbf{C}_n$ denote the $2^n \times 2^n \times 2^n$ cube, $\widetilde{\mathbf{C}_n}$ the corner-punctured cube, and let $\mathbf{C}'_n$ be $\mathbf{C}_n$ but with an arbitrary ***cubie*** removed.

What is the 3-dimensional analog of an Lmino? Calling it a "**3-*mino***", how many cubies does it have? [Provide a drawing of your 3-mino.] PROVE: *Every $\mathbf{C}'_n$ admits a 3-mino–tiling.* [Provide also pictures showing your ideas.]

iv) Generalize to K-dim(ensional) space, with $\mathbf{C}_{n,K}$ being the $2^n \times .\overset{K}{.}. \times 2^n$ cube, having $[2^n]^K = 2^{nK}$ many K-dim'al cubies. As before, let $\mathbf{C}'_{n,K}$ be $\mathbf{C}_{n,K}$ with an arbitrary cubie removed.

*What is your K-**mino** with which you will tile, and how many cubies does it have?* (So a 2-mino is our Lmino.) PROVE: *Every $\mathbf{C}'_{n,K}$ admits a K-mino–tiling.*

A3: Recall *Rabbits and Lights* from the zoomester's beginning: To your right are lights $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \ldots$. Each light has a toggle button; Press&release: the light illuminates; P&R again, it is extinguished.

Off to your left is a queue of rabbits; so we have

$$\ldots \mathcal{R}_3\ \mathcal{R}_2\ \mathcal{R}_1\quad \mathcal{L}_1\ \mathcal{L}_2\ \mathcal{L}_3\ \mathcal{L}_4, \ldots$$

All the lights are initially off. If rabbit-α (i.e, $\mathcal{R}_\alpha$) jumps, then he will hop on lights $\mathcal{L}_\alpha, \mathcal{L}_{2\alpha}, \mathcal{L}_{3\alpha}, \ldots$, turning them all on. If rabbit-β now jumps, he will change the state of lights $\beta, 2\beta, 3\beta, \ldots$, turning some on, and some off.

A Map f. A (finite or infinite) set $\mathsf{R} = \{\alpha_1, \alpha_2, \ldots\}$ of rabbit-indices is an element of powerset $\mathcal{P} := \mathcal{P}(\mathbb{Z}_+)$. After those rabbits jump, we have a (finite or infinite) set $\mathsf{L} = \{\beta_1, \beta_2, \beta_2, \ldots\}$ of indices of illuminated lights. Define $f : \mathcal{P} \to \mathcal{P}$ by $f(\mathsf{R}) := \mathsf{L}$.

Our first-day class showed [*involution argument, and re-argued using the divisor-count τ-fnc*] that $f(\mathbb{Z}_+)$ is the set $\{1, 4, 9, \ldots\}$ of squares. Evidently $f(\varnothing) = \varnothing$ and $f(\{1, 2\}) = \text{Odds}$. □

Q1) For each of the following questions, produce *either* a CEX [counterexample] *or* a formal proof.
Is f injective? Is f surjective?

Q2 For $\mathsf{L} \in \text{Range}(f)$, give an algorithm to produce an R for which $f(\mathsf{R}) = \mathsf{L}$. If you program, can you implement your algorithm in computer code?

Q3 Produce a commutative, associative binop $\$: \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ which satisfies

$$\forall \mathsf{R}, \mathsf{R}' : \quad f\big(\mathsf{R} \ \$ \ \mathsf{R}'\big) \ = \ f\big(\mathsf{R}\big) \ \$ \ f\big(\mathsf{R}'\big) \,.$$

What can you tell me about this binary operator?

Q4 What is the f–fixed-point set; those R with $f(\mathsf{R}) = \mathsf{R}$?

What can you say about the dynamics of f? —does it have periodic points of order 2? 3? ...?

What is $f\big(f(\mathbb{Z}_+)\big) \stackrel{\text{note}}{=\!=} f\big(\text{Squares}\big)$? (Conjecture? Computer simulation?)

Q5 Creativity: Come up with an *interesting* generalization of this problem. E.g:

- What if each light has *three* states; off, dim, bright?
- What if we have a plane, or quarter-plane of lights, and interesting rule for rabbits jumping on it?

Can you give a (partial) solution, or a computer simulation, or a conjecture, for your new problem(s)?

A1:	___ ___	50pts
A2:	___ ___ ___	125pts
A3:	___ ___ ___	130pts
Total:	___ ___ ___	305pts

HONOR CODE: *"I have neither requested nor received help on this exam other than from my team-mates and my professor (or his colleague)."* *Name/Signature/Ord*

Ord: ....

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